THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
Exercise for MATH1010 University Mathematics

Contents

1 Preliminaries

1.1 Functions

1. Graph the functions $f(x) = \frac{x}{2}$ and $g(x) = \frac{4}{x} - 1$ together, to identify values of x for which
$$\frac{x}{2} > \frac{4}{x} - 1.$$
Confirm your answer by solving the inequality algebraically.

2. Plot all points (x, y) on the plane that satisfy $x^2 + y^2 + 2x - 4y - 14 = 0$. Explain why it is not the graph of any function.

3. Find all real numbers x satisfying
$$2^x + 2^{1-x} = 3.$$

4. Determine whether each of the following statement is true or false.
   (a) $e^{x+y} = e^x + e^y$ for all real numbers x and y
   (b) $\ln(x + y) = (\ln x)(\ln y)$ for all $x, y > 0$
   (c) $e^{xy} = e^x + e^y$ for all real numbers x and y
   (d) $\ln(x^y) = (\ln x)^y$ for all $x, y > 0$
   (e) If $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g'(x)$.
   (f) $\frac{d}{dx}(2^x) = x2^{x-1}$
   (g) $\int \ln x \, dx = \frac{1}{x} + C$

1.2 Limits and Derivatives

1. Evaluate the following limits
   (a) $\lim_{x\to 4} \frac{2 - \sqrt{x}}{4x - x^2}$
   (b) $\lim_{x\to 0} \frac{x^{928}}{\sqrt{x^{928} + 4} - 2}$
   (c) $\lim_{t\to 0} \left(\frac{1}{2t} - \frac{1}{t^2 + 2t} \right)$
   (d) $\lim_{t\to 0} \frac{\ln(e + t)}{et}$
   (e) $\lim_{x\to -2} \frac{|x|^3 - 8}{x^4 - 16}$

2. Use first principles to find $\frac{dy}{dx}$ of the following functions.
   (a) $y = x^3 - 4x$
   (b) $y = 3 - 2\sqrt{x}$
   (c) $y = \frac{6}{x^2}$
   (d) $y = \frac{1}{\sqrt{x}}$

3. Find the first derivative of the following functions.

   (a) $f(x) = \ln(x^\pi + e^\pi)$

   (b) $g(x) = \dfrac{1}{\sqrt{\pi^x + x^4}}$

   (c) $h(x) = xe^{1/x^2} + \dfrac{\ln(2x)}{\sqrt{x}}$

   (d) $u(x) = \dfrac{x}{\sqrt{e^{-x} + \sqrt{x}}}$

   (e) $v(x) = \ln \dfrac{(1+x^2)^{87}}{x^2}$

4. If $t = (x+1)(x+2)^2(x+3)^3$ and $\ln t = e^y$, find $\dfrac{dy}{dx}$ in terms of x only.

5. Suppose $f'(2) = 13$, $g(7) = 2$ and $g'(7) = 53$. If $y = f(g(x))$, find $\left.\dfrac{dy}{dx}\right|_{x=7}$.

6. Suppose $y = f(x)$ is a smooth function. Determine whether each of the following statements is true or false.

   (a) If f is increasing, then $f''(x) > 0$ for all x.

   (b) If $f'(x) > 0$ for all x, then $f(x) > 0$ for all x.

   (c) If $f'(x) > 0$ for all x, then f' is increasing.

   (d) If $f(x) > 0$ for all x, then $f'(x) > 0$ for all x.

   (e) If $f'(3) = 0$, then f must have a maximum or a minimum at $x = 3$.

1.3 Applications of Derivatives

1. If L is a tangent line to the graph $y = x^2 + 3$, and the x-intercept of L is 1, find all possible points at which L touches the graph of $y = x^2 + 3$.

2. The position x of a particle at time t is given by

$$x = t^3 + at^2 + bt + c$$

   for some constants a, b, c. It is known that when $t = 0$, the particle is at position $x = 0$; also, there is a certain time t_0 such that the velocity and acceleration of the particle are both zero at time t_0, and at time t_0 the particle is at position $x = 1$. Find the values of a, b and c.

3. Water is being drained, at a constant rate of 1/4 cubic meter per hour, from the bottom of a container that takes the shape of an inverted regular cone. The radius of the base of the cone is 3 meters, and the height of the cone is 2 meters. What is the rate of change of the depth of the water, when the water is 1 meter deep?

4. Let C be the curve $y = \dfrac{1}{x} + x$. Note that $P = (1, 2)$ and $Q = (0.5, 2.5)$ are two points on C.

   (a) Find equations of the tangent and normal to C at the point P.

   (b) Show that the tangent to C at the point Q passes through the point $A = (0, 4)$.

5. A water container is made in the shape of a right circular cone, with its axis perpendicular to the ground and its apex at the bottom of the container. The axis subtends an angle of $30°$ with the (conical) surface of the container. Water is flowing out of the cone through the apex at a constant rate of π cm^3 per second.

   (a) Suppose that when the depth of water is h cm, the volume of water in the container is V cm^3. Express V in terms of h.

   (b) How fast is the water level falling when the depth of water is 4 cm?

6. Let C be the curve given by the equation $y = \dfrac{x^2}{1+x} - \dfrac{4}{3}$.

   (a) Find the x- and y-intercepts of the curve C.

   (b) Find $\dfrac{dy}{dx}$ and show that $\dfrac{d^2y}{dx^2} = \dfrac{a}{(1+x)^b}$, where a, b are integers whose respective values you have to determine.

   (c) Find the turning point(s) of the curve C.
   For each turning point, determine whether it is a (relative) maximum or a (relative) minimum point.

   (d) What are the asymptotes of the curve C, if any?

   (e) Sketch the curve C for $-1 < x < 3$.

7. The function $f(x) = \dfrac{x^2 + kx + 9}{x^2 + 1}$, where k is a constant, attains a stationary value at $x = 3$.

   Consider the curve C given by $y = f(x)$.

   (a) Determine the value of k.

   (b) i. Find the x- and y-intercepts of the curve $y = f(x)$.
   ii. Find the (relative) maximum and (relative) minimum points of the curve C.
   iii. What are the asymptotes of the curve C, if any?

   (c) Sketch the curve C for $-6 \le x \le 6$.

8. Let Σ be a sphere with (fixed) radius R.

   (a) Suppose Γ be a right circular cylinder inscribed in the sphere Σ, with height h, radius r, volume V and surface area A.
   i. Express V in terms of r and R alone.
   ii. Express A in terms of r and R alone.

   (b) What is the largest possible value of V?

   (c) What is the largest possible value of A?

9. Let $\beta \in (1, +\infty)$. Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^\beta + \beta - 1 - \beta x$ for any $x \in (0, +\infty)$.

(a) i. Compute f'.
 ii. Show that f is strictly decreasing on $(0, 1]$.
 iii. Show that f is strictly increasing on $[1, +\infty)$.
 iv. Determine whether f attains the maximum and/or the minimum on $(0, +\infty)$.

(b) Hence, or otherwise, show that $(1 + r)^\beta \geq 1 + \beta r$ for any $r \in (-1, +\infty)$.

1.4 Integration

1. Evaluate the following indefinite integrals.

   (a) $\displaystyle\int (3 - x^2)^3 dx$

   (b) $\displaystyle\int x^2(5 - x)^4 dx$

   (c) $\displaystyle\int \frac{x + 1}{\sqrt{x}}\, dx$

   (d) $\displaystyle\int \left(8t - \frac{2}{t^{\frac{1}{4}}}\right) dt$

2. Use a suitable substitution to evaluate the following integral.

   (a) $\displaystyle\int \frac{dx}{\sqrt{2 - 5x}}$

   (b) $\displaystyle\int \frac{e^{3x} + 1}{e^x + 1}\, dx$

   (c) $\displaystyle\int \frac{x}{\sqrt{1 - x^2}}\, dx$

   (d) $\displaystyle\int x^2 \sqrt[3]{1 + x^3}\, dx$

   (e) $\displaystyle\int \frac{x dx}{(1 + x^2)^2}$

   (f) $\displaystyle\int xe^{-x^2} dx$

   (g) $\displaystyle\int \frac{e^x dx}{2 + e^x}$

   (h) $\displaystyle\int \frac{dx}{1 + e^x}$

3. Evaluate the following definite integrals.

   (a) $\displaystyle\int_1^3 \frac{2x^3 - 5}{x^2} dx$

   (b) $\displaystyle\int_0^1 x\sqrt{1 - x^2} dx$

   (c) $\displaystyle\int_0^1 \frac{5x}{(4 + x^2)^2} dx$

   (d) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} dx$

4. Find the area of the regions bounded by the graphs of the given functions.

   (a) $y = 4 - x^2$; x-axis

   (b) $y = 3 - x^2$; $y = -x - 3$

   (c) $y = x^2 - 4$; $y = -x^2 - 2x$

   (d) $y = x^2$; $y = x - 2$; x-axis

   (e) $y = \sqrt{x}$; $y = x - 2$; x-axis

   (f) $x + y^2 = 4$; $x + y = 2$

5. The slope at any point (x, y) of a curve C is given by $\dfrac{dy}{dx} = 4 - 2x$ and C passes through the point $(1, 0)$.

   (a) Find an equation of the curve C.

   (b) Find the area of the finite region bounded by the curve C and the x-axis.

6. The slope of the tangent to a curve C at any point (x, y) on C is $x^2 - 2$. C passes through the point $(3, 4)$.

(a) Find an equation of the curve C.

(b) Find the coordinates of the point on C at which the slope of the tangent is -2.

7. The curve $y = x^3 - x^2 - 2x$ cuts the x-axis at the origin and the points $(a, 0)$, $(b, 0)$, where $a < 0 < b$.

   (a) Find the values of a and b.

   (b) Find the total area of the region bounded by the curve and the x-axis.

8. The slope at any point (x, y) of a curve C is given by $\dfrac{dy}{dx} = 6x + \dfrac{1}{x^2}$, where $x > 0$.

   Suppose the curve C cuts the x-axis at the point $(1, 0)$. Find its equation.

2 Differentiation

2.1 Graphs of Functions

1. Sketch the graphs of the following functions.

(a) $f(x) = x^2 - 4x - 5$

(b) $f(x) = x^3 - 4x$

(c) $f(x) = x^3 + x^2 - 5x + 3$

(d) $f(x) = x + \dfrac{1}{x}$

(e) $f(x) = \dfrac{2x-9}{x+3}$, $x \neq -3$

(f) $f(x) = \dfrac{x^2}{x-2}$, $x \neq 2$

(g) $f(x) = 3 - \sqrt{4-x^2}$, $-2 \leq x \leq 2$

(h) $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

(i) $f(x) = 3 - e^x$

(j) $f(x) = \dfrac{e^x - e^{-x}}{2}$

(k) $f(x) = 2 + \ln(x+1)$, $x > -1$

(l) $f(x) = 5 - \ln(x-2)^2$, $x \neq 2$

(m) $f(x) = 1 - 2\sin x$

(n) $f(x) = 2 + \cos 2x$

(o) $f(x) = 3\cos\left(x + \frac{\pi}{3}\right)$

(p) $f(x) = \cos x - \sin x$

(q) $f(x) = 1 + \sin^2 x$

(r) $f(x) = 1 + \cos x \sin x$

(s) $f(x) = x\cos x$

2. Sketch the graphs of the following functions.

(a) $f(x) = |x^2 - 2x - 3|$

(b) $f(x) = x^2 - 4|x| + 3$

(c) $f(x) = ||x-2| - 4|$

(d) $f(x) = |3 - |x^2 - 1||$

(e) $f(x) = |2 + 3\sin x|$

(f) $f(x) = 2 - |\sin x|$

3. Sketch the graphs of the following piecewise defined functions.

(a) $f(x) = \begin{cases} 2x+5, & x < -1 \\ x^2 - 1, & x \geq -1 \end{cases}$

(b) $f(x) = \begin{cases} x, & x < -2 \\ -x, & -2 \leq x < 2 \\ x, & x \geq 2 \end{cases}$

(c) $f(x) = \begin{cases} 1 - x^2, & x < 1 \\ \ln x, & x \geq 1 \end{cases}$

(d) $f(x) = \begin{cases} 1 - |x+3|, & x < -2 \\ 2 - |x|, & -2 \leq x < 2 \\ 1 - |x-3|, & x \geq 2 \end{cases}$

2.2 Limits and Continuity

1. Evaluate the following limits

(a) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{x^2}$

(b) $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{4x}$

(c) $\displaystyle\lim_{x\to 0} \frac{x^2}{1-\cos x}$

(d) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x}$

(e) $\lim_{x\to 0} \frac{\sin 9x - \sin 3x}{\sin 2x}$

(f) $\lim_{x\to 0} \frac{\cos x - \cos 3x}{x^2}$

(g) $\lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x}$

(h) $\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3}$

(i) $\lim_{x\to \frac{\pi}{4}} \tan 2x \tan\left(\frac{\pi}{4} - x\right)$

(j) $\lim_{x\to \frac{\pi}{3}} \frac{\sin(x - \frac{\pi}{3})}{1 - 2\cos x}$

(k) $\lim_{t\to 0} \frac{\ln(1+t)}{t}$

(l) $\lim_{h\to 0} \frac{e^h - 1}{h}$

2. Evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{2x+5}{x^2+3}$

(b) $\lim_{x\to\infty} \frac{3x+1}{5x-4}$

(c) $\lim_{x\to\infty} \frac{6x^2+2x-5}{2x^2-4x+1}$

(d) $\lim_{x\to\infty} \frac{2x^3-5x+3}{3x^3-x^2+6}$

(e) $\lim_{x\to+\infty} \frac{\sqrt{9x^4-3x+2}}{x^2-2x+5}$

(f) $\lim_{x\to+\infty} \left(\sqrt{4x^2+5x} - 2x\right)$

(g) $\lim_{x\to-\infty} \left(x + \sqrt{x^2+4x}\right)$

(h) $\lim_{x\to-\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2-x+1}\right)$

(i) $\lim_{x\to+\infty} \frac{\sqrt{9x^4-3x+2}}{x^2-2x+5}$

(j) $\lim_{x\to-\infty} \left(x + \sqrt{x^2+4x}\right)$

(k) $\lim_{x\to+\infty} \frac{8e^{-9x}+x^2}{6e^{4x}-x^2}$

(l) $\lim_{x\to+\infty} \frac{e^x+x^2}{e^x-x^2}$

(m) $\lim_{x\to+\infty} \frac{\ln(x^2+x-1)}{\ln(x^8-x+1)}$

(n) $\lim_{x\to+\infty} \frac{\ln(x^2+e^{3x})}{\ln(x^3+e^x)}$

(o) $\lim_{x\to+\infty} \frac{\ln(1+8^x)}{\ln(1+2^x)}$

(p) $\lim_{x\to-\infty} \frac{\ln(1+8^x)}{\ln(1+2^x)}$

(q) $\lim_{x\to\infty} \frac{\sin x}{x}$

(r) $\lim_{x\to+\infty} \frac{\cos x}{(\ln x)^2}$

2.3 Derivatives

1. For each of the following functions, determine whether it is differentiable at $x = 0$. Find $f'(0)$ if it is.

(a) $f(x) = x^{\frac{4}{3}}$

(b) $f(x) = |\sin x|$

(c) $f(x) = x|x|$

(d) $f(x) = \begin{cases} 5 - 2x, & \text{when } x < 0 \\ x^2 - 2x + 5, & \text{when } x \geq 0 \end{cases}$

(e) $f(x) = \begin{cases} 1 + 3x - x^2, & \text{when } x < 0 \\ x^2 + 3x + 2, & \text{when } x \geq 0 \end{cases}$

(f) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

2. Find all values of a and b for which the function

$$f(x) = \begin{cases} \sin x, & \text{for } x < \pi \\ ax + b, & \text{for } x \geq \pi \end{cases}$$

is differentiable at $x = \pi$.

3. Let

$$f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2}, & \text{for } x < 0 \\ ax + b, & \text{for } x \geq 0 \end{cases}$$

Find the value of a and b if $f(x)$ is differentiable at $x = 0$.

4. Find the first derivatives of the following functions.

(a) $y = x^3 - 4x + 3$
(b) $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$
(c) $y = x^2 e^{5x}$
(d) $y = \cos x \ln x$
(e) $y = \sin x \cos x$
(f) $y = 3 \sec x - \tan x$
(g) $y = x \cot x$
(h) $y = \dfrac{3x - 4}{x + 2}$
(i) $y = \dfrac{x^2 + 1}{x + 1}$
(j) $y = \dfrac{\sin x}{x}$
(k) $y = \dfrac{\tan x}{\sqrt{x}}$
(l) $y = (x^2 + 1)^7$
(m) $y = \sqrt{x^4 + 1}$
(n) $y = \cos(x^2)$
(o) $y = xe^{x^3 + x}$
(p) $y = \ln(\ln x)$
(q) $y = e^{\sin x}$
(r) $y = \dfrac{x}{\sqrt{1 + x^2}}$
(s) $y = \ln(x + \sqrt{1 + x^2})$
(t) $y = \sqrt{x + \sqrt{x}}$
(u) $y = \cosh^2 x$
(v) $y = \dfrac{\sinh^2 x}{\cosh x}$
(w) $y = \ln(\sinh x)$
(x) $y = \sin^{-1} \sqrt{x}$
(y) $y = \cos(\tan^{-1} x)$

5. Find the first derivatives of the following functions.

(a) $y = 3^x$
(b) $y = 2^{\cos x}$
(c) $y = x^x$
(d) $y = x^{\sqrt{x}}$
(e) $y = (\cos x)^{\sin x}$
(f) $y = x^{x^x}$

6. Find $\dfrac{dy}{dx}$ for the following implicit functions.

(a) $x^2 + y^2 = 4$
(b) $x^3 y + xy^2 = 1$
(c) $x^3 + y^3 = 2xy$
(d) $xe^{xy} = 1$
(e) $\sin(xy) = (x + y)^2$
(f) $\cos\left(\dfrac{y}{x}\right) = \ln(x + y)$

7. Find $\dfrac{d^2y}{dx^2}$ for the following functions.

(a) $y = \sqrt{x} e^{x^2}$
(b) $y = \dfrac{x}{\sqrt{1 + x^2}}$
(c) $y = (\ln x)^2$
(d) $y = \sec x$
(e) $y = \tan^{-1} x$
(f) $x^2 + y^3 = 1$
(g) $x^4 y - 3x^2 y^3 = 5$
(h) $y = 3^{x^2}$
(i) $y = x^{\ln x}$

8. Prove that the Chebyshev polynomials

$$T_m(x) = \frac{1}{2^{m-1}} \cos(m \cos^{-1} x), \; m = 0, 1, 2, \cdots$$

satisfy

$$(1 - x^2)T''_m(x) - xT'_m(x) + m^2 T_m(x) = 0$$

9. Prove that the Legendre polynomials

$$P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m}(x^2 - 1)^m \; m = 0, 1, 2, \cdots$$

satisfy

$$(1 - x^2)P''_m(x) - 2xP'_m(x) + m(m + 1)P_m(x) = 0$$

10. Show that the function

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is not differentiable at $x = 0$.

11. Show that the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is differentiable at $x = 0$ but $f'(x)$ is not continuous at $x = 0$.

2.4 Mean Value Theorem and Taylor's Theorem

1. Using the mean value theorem to prove for $0 < y < x$ and $p > 1$,

$$py^{p-1}(x - y) < x^p - y^p < px^{p-1}(x - y).$$

2. Using the mean value theorem to prove that for $0 \leq x_1 < x_2 < x_3 \leq \pi$,

$$\frac{\sin x_2 - \sin x_1}{x_2 - x_1} > \frac{\sin x_3 - \sin x_2}{x_3 - x_2}.$$

3. Using the mean value theorem to prove that for $x > 0$,

$$\frac{x}{1 + x} < \ln(1 + x) < x.$$

Hence, deduce that for $x > 0$,

$$\frac{1}{1 + x} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}.$$

4. By applying the mean value theorem, prove that the equation

$$a_1x + a_2x^2 + \cdots + a_nx^n = \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1}$$

has a root between 0 and 1.

5. Let $f(x)$ be a function defined on $[0, \infty)$ such that

- $f(0) = 0$,
- $f'(x)$ exists and is monotonic increasing on $(0, \infty)$.

Prove that

$$f(a + b) \leq f(a) + f(b)$$

for $0 \leq a \leq b \leq a + b$.

6. Let A be a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Suppose that there exists $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for any $x, y \in A$, then f is said to be satisfying the **Lipschitz condition** on A.

Prove that $\sin x$ satisfies the Lipschitz condition on $\mathbb{R}$.

7. Let $f(x)$ be a function such that $f'(x)$ is strictly decreasing for $x > 0$.

(a) Using the mean value theorem, show that

$$f'(k + 1) < f(k + 1) - f(k) < f'(k) \text{ and } k \geq 1.$$

(b) Using (a), prove that for any integer $n \geq 2$,

$$f'(2) + f'(3) + \cdots + f'(n) < f(n) - f(1) < f'(1) + f'(2) + \cdots + f'(n - 1).$$

8. Find the Taylor polynomials of the given orders for the following functions at $x = 0$.

(a) $\ln(1 + x)$, order $= 4$

(b) $\cosh x = \frac{e^x + e^{-x}}{2}$, order $= 6$

(c) $\sinh x = \frac{e^x - e^{-x}}{2}$, order $= 5$

9. Find the Taylor Series for the following functions at the given points.

(a) $\frac{1}{1+x}$ at $x = 0$

(b) $\sin x$ at $x = \dfrac{\pi}{2}$.

(c) e^x at $x = 1$.

10. (a) Write down the Taylor polynomial $P_3(x)$ of degree 3 generated by $f(x) = \ln(1 - x)$ at 0

(b) Hence approximate $\ln 0.99$.

11. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function that can be differentiated infinitely many times and the Taylor series expansion of f at $x = 0$ is

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

   (a) If f is an odd function, show that $a_0 = a_2 = a_4 = \cdots = 0$.

   (b) If f is an even function, show that $a_1 = a_3 = a_5 = \cdots = 0$.

2.5 L'Hopital's Rule

1. Use L'Hopital's rule to evaluate the following limits.

   (a) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 5x}$

   (b) $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{1-\cos x}$

   (c) $\lim\limits_{x\to 0} \dfrac{2\sin x - \sin 2x}{x - \sin x}$

   (d) $\lim\limits_{x\to 0} \dfrac{1 - x\cot x}{x \sin x}$

   (e) $\lim\limits_{x\to 0} \dfrac{\sinh x - \sin x}{x(\cosh x - \cos x)}$

   (f) $\lim\limits_{x\to 0} \dfrac{\ln\cos 2x}{\ln\cos x}$

   (g) $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$

   (h) $\lim\limits_{x\to 1} \left(\dfrac{1}{\ln x} - \dfrac{1}{x - 1}\right)$

   (i) $\lim\limits_{x\to 0} \dfrac{e^x - x - 1}{\cosh x - 1}$

   (j) $\lim\limits_{x\to 0} \dfrac{e - (1+x)^{\frac{1}{x}}}{x}$

   (k) $\lim\limits_{x\to 0} \dfrac{2^x - 1}{x}$

   (l) $\lim\limits_{x\to 0^+} x^{\frac{1}{1+\ln x}}$

   (m) $\lim\limits_{x\to 1} x^{\frac{1}{1-x}}$

   (n) $\lim\limits_{x\to +\infty} \dfrac{\ln(2x^3 - 5x^2 + 3)}{\ln(4x^2 + x - 7)}$

   (o) $\lim\limits_{x\to +\infty} x \sin\left(\dfrac{1}{x}\right)$

   (p) $\lim\limits_{x\to +\infty} x\left(\dfrac{\pi}{2} - \tan^{-1} x\right)$

   (q) $\lim\limits_{x\to +\infty} x \ln\left(1 + \dfrac{3}{x}\right)$

   (r) $\lim\limits_{x\to +\infty} (e^x + x)^{\frac{1}{x}}$

3 Integration

3.1 Fundamental Theorem of Calculus

1. Find $F'(x)$ for the following functions.

(a) $F(x) = \int_{\pi}^{x} \frac{\cos y}{y}\, dy$

(b) $F(x) = \int_{-\pi}^{x} e^{\sin 2t}\, dt$

(c) $F(x) = \int_{x}^{1} \sqrt{1+t^2}\, dt$

(d) $F(x) = \int_{0}^{x^3} e^{u^2}\, du$

(e) $F(x) = \int_{-\sin x}^{\sqrt{\pi}} \cos(y^2)\, dy$

(f) $F(x) = \int_{x}^{2x} (\ln t)^2\, dt$

(g) $F(x) = \int_{x^2}^{x^3} e^{\cos u}\, du$

(h) $F(x) = \int_{-\sqrt{\ln x}}^{\sqrt{\ln x}} \frac{\sin t}{t}\, dt$

3.2 Substitution

Use a suitable substitution to evaluate the following integral.

1. $\int \frac{dx}{\sqrt{2-5x}}$
2. $\int \frac{e^{3x}+1}{e^x+1}\, dx$
3. $\int \frac{x}{\sqrt{1-x^2}}\, dx$
4. $\int x^2 \sqrt[3]{1+x^3}\, dx$
5. $\int \frac{x dx}{(1+x^2)^2}$
6. $\int \frac{dx}{\sqrt{x}(1+x)}$
7. $\int \frac{1}{x^2} \sin \frac{1}{x}\, dx$
8. $\int x e^{-x^2} dx$
9. $\int \frac{e^x dx}{2+e^x}$
10. $\int \frac{dx}{e^x+e^{-x}}$
11. $\int \tan x dx$
12. $\int \frac{dx}{1+e^x}$
13. $\int x(x^2+2)^{99} dx$
14. $\int \frac{x}{\sqrt{25-x^2}} dx$
15. $\int \frac{x}{\sqrt{3x^2+1}} dx$
16. $\int \frac{x^2}{\sqrt{9-x^3}} dx$
17. $\int x(x+2)^{99} dx$
18. $\int \frac{x dx}{\sqrt{4x+5}}$
19. $\int x\sqrt{x-1} dx$
20. $\int (x+2)\sqrt{x-1} dx$

21. $\displaystyle\int \frac{xdx}{\sqrt{x+9}}$

22. $\displaystyle\int_0^1 x^3(1+3x^2)^{\frac{1}{2}}dx$

3.3 Integration by Parts

1. $\displaystyle\int \ln x dx$
2. $\displaystyle\int x^2 \ln x dx$
3. $\displaystyle\int \left(\frac{\ln x}{x}\right)^2 dx$
4. $\displaystyle\int xe^{-x}dx$
5. $\displaystyle\int x^2e^{-2x}dx$
6. $\displaystyle\int x\cos x dx$
7. $\displaystyle\int x^2 \sin 2x dx$
8. $\displaystyle\int \sin^{-1} x dx$
9. $\displaystyle\int x\tan^{-1} x dx$
10. $\displaystyle\int \ln(x+\sqrt{1+x^2})dx$
11. $\displaystyle\int x\sin^2 x dx$
12. $\displaystyle\int \sin(\ln x)dx$
13. $\displaystyle\int x\sin(4x)dx$
14. $\displaystyle\int x\cos(5-x)dx$
15. $\displaystyle\int \cos^{-1}(x)dx$
16. $\displaystyle\int x\cos^{-1}(x)dx$
17. $\displaystyle\int \tan^{-1}(x)dx$
18. $\displaystyle\int x\tan^{-1}(x)dx$
19. $\displaystyle\int \ln^2(x)dx$
20. $\displaystyle\int x^{99}\ln(x)dx$
21. $\displaystyle\int \frac{\ln(x)}{x^{101}}dx$
22. $\displaystyle\int x\sec^2(x)dx$
23. $\displaystyle\int e^{2x}\cos(3x)dx$

3.4 Reduction Formula

Prove the following reduction formulas.

1. $\displaystyle I_n = \int x^n e^{ax}dx;\ I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a}I_{n-1},\ n \geq 1$
2. $\displaystyle I_n = \int \sin^n x dx;\ I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}I_{n-2},\ n \geq 2$

3. $I_n = \int \cos^n x dx$; $I_n = \dfrac{\sin x \cos^{n-1} x}{n} + \dfrac{n-1}{n} I_{n-2}$, $n \geq 2$

4. $I_n = \int \dfrac{1}{\sin^n x} dx$; $I_n = -\dfrac{\cos x}{(n-1)\sin^{n-1} x} + \dfrac{n-2}{n-1} I_{n-2}$, $n \geq 2$

5. $I_n = \int x^n \cos x dx$; $I_n = x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}$, $n \geq 2$

6. $I_n = \int \dfrac{dx}{(x^2 - a^2)^n}$; $I_n = -\dfrac{x}{2a^2(n-1)(x^2-a^2)^{n-1}} + \dfrac{2n-3}{2a^2(n-1)} I_{n-1}$, $n \geq 1$

7. $I_n = \int \dfrac{x^n dx}{\sqrt{x+a}}$; $I_n = \dfrac{2x^n\sqrt{x+a}}{2n+1} - \dfrac{2an}{2n+1} I_{n-1}$, $n \geq 1$

8. $I_n = \int (\ln x)^n dx$; $I_n = x(\ln x)^n - n I_{n-1}$, $n \geq 1$.

9. $I_n = \int_0^1 x^n \sqrt{1-x} dx$; $I_n = \dfrac{2n}{2n-3} I_{n-1}, n \geq 2$.

3.5 Trigonometric Integrals

Evaluate

1. $\int \dfrac{dx}{1 - \cos x}$
2. $\int \sin^5 x \cos x dx$
3. $\int \sin 3x \sin 5x dx$
4. $\int \cos \dfrac{x}{2} \cos \dfrac{x}{3} dx$
5. $\int \cos^3 x dx$
6. $\int \sin^4 x dx$
7. $\int \dfrac{dx}{\cos x \sin^2 x}$
8. $\int \dfrac{\sin x \cos^3 x}{1 + \cos^2 x}, dx$
9. $\int \tan^5 x dx$
10. $\int \dfrac{dx}{\sin^4 x \cos^4 x}, dx$
11. $\int \sin 5x \cos x dx$
12. $\int \cos x \cos 2x \cos 3x dx$
13. $\int \cos^5 x \sin^3 x dx$
14. $\int \cos^5 x \sin^4 x dx$
15. $\int \sin^2 x \cos^4 x dx$

3.6 Trigonometric Substitution

Evaluate the following integrals by trigonometric substitution.

1. $\int \frac{x^2}{1+x^2}\,dx$
2. $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}$
3. $\int \sqrt{\frac{1+x}{1-x}}\,dx$
4. $\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$
5. $\int \frac{x^2dx}{\sqrt{9-x^2}}$
6. $\int \frac{dx}{\sqrt{4+x^2}}$
7. $\int x^2\sqrt{16-x^2}dx$
8. $\int \frac{dx}{x^2\sqrt{x^2+4}}$
9. $\int \frac{dx}{(4x^2+1)^{3/2}}$
10. $\int \frac{1}{(2x-x^2)^{3/2}}$

3.7 Rational Functions

Evaluate the following integrals of rational functions.

1. $\int \frac{x^2dx}{1-x^2}$
2. $\int \frac{x^3}{3+x}\,dx$
3. $\int \frac{(1+x)^2}{1+x^2}\,dx$
4. $\int \frac{dx}{x^2+2x-3}$
5. $\int \frac{dx}{(x^2-2)(x^2+3)}$
6. $\int \frac{x^2+1}{(x+1)^2(x-1)},dx$
7. $\int \frac{x^2}{(x^2-3x+2)^2},dx$
8. $\int \frac{x^2+5x+4}{x^4+5x^2+4},dx$
9. $\int \frac{dx}{(x+1)(x^2+1)}$
10. $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$
11. $\int \frac{4-2x}{(x^2+1)(x-1)^2}\,dx$
12. $\int \frac{dx}{x(x^2+1)^2}$
13. $\int \frac{x^2dx}{(x-1)(x-2)(x-3)}$
14. $\int \frac{xdx}{x^2(x^2-2x+2)}$

3.8 *t*-method

Use t-substitution to evaluate the following integrals.

1. $\int \frac{dx}{\sin^3 x}$
2. $\int \frac{dx}{1+\sin x}$
3. $\int \frac{dx}{\sin x\cos^4 x}$
4. $\int \frac{dx}{2+\sin x}$

5. $\int \frac{1-\cos x}{3+\cos x} dx$

6. $\int \frac{\cos x + 1}{\sin x + \cos x} dx$

3.9 Piecewise Functions

Find $\int f(x)dx$ for the following functions $f(x)$.

1. $f(x) = \begin{cases} 4x-1, & x < 1 \\ \frac{3}{\sqrt{x}}, & x \geq 1 \end{cases}$

2. $f(x) = \begin{cases} e^{2x}, & x < 0 \\ \cos^2 x, & x \geq 0 \end{cases}$

3. $f(x) = |x-3|$

4. $f(x) = \frac{1}{1+|x|}$

5. $f(x) = 3|x^2 - 4|$

6. $f(x) = |\ln x|$

4 Further Problems

1. (a) Evaluate $\lim_{x\to 0} \dfrac{x \sin x}{1-\cos x}$.

   (b) Using (a), evaluate $\lim_{x\to 0^+} (1-\cos x)^{\frac{1}{\ln x}}$.

2. Let f and g be differentiable functions defined on $\mathbb{R}$ satisfying the following conditions:

   - $f'(x) = g(x)$ for $x \in \mathbb{R}$;
   - $g'(x) = -f(x)$ for $x \in \mathbb{R}$;
   - $f(0) = 0$ and $g(0) = 1$.

   By differentiating $h(x) = [f(x) - \sin x]^2 + [g(x) - \cos x]^2$, show that $f(x) = \sin x$ and $g(x) = \cos x$ for $x \in \mathbb{R}$.

3. Let $a > b > 0$ and define

$$f(x) = \begin{cases} \left(\dfrac{a^x + b^x}{2}\right)^{1/x} & \text{for} \quad x > 0, \\ \sqrt{ab} & \text{for} \quad x = 0. \end{cases}$$

   (a) (i) Evaluate $\lim_{x\to 0^+} f(x)$.
   Hence show that f is continuous at $x = 0$.

   (ii) Show that $\lim_{x\to\infty} = a$.

   (b) Let $h(t) = (1+t)\ln(1+t) + (1-t)\ln(1-t)$ for $0 \le t < 1$ and $g(x) = \ln f(x)$ for $x \ge 0$.

   (i) Show that $h(t) > h(0)$ for $0 < t < 1$.

   (ii) For $x > 0$, let $t = \dfrac{a^x - b^x}{a^x + b^x}$. Show that $0 < t < 1$ and

$$h(t) = 2\left[\frac{a^x \ln a^x + b^x \ln b^x}{a^x + b^x} + \ln\left(\frac{2}{a^x + b^x}\right)\right].$$

   (iii) Show that for $x > 0$,

$$x^2 g'(x) = \frac{a^x \ln a^x + b^x \ln b^x}{a^x + b^x} + \ln\left(\frac{2}{a^x + b^x}\right).$$

   Hence deduce that $f(x)$ is strictly increasing on $[0, \infty)$.

4. Let f be real-valued function on $[0, 1]$ and differentiable in $(0, 1)$. Suppose f satisfies

   - $f(0) = 0$;
   - $f(1) = 1/2$;
   - $0 < f'(t) < 1$ for $t \in (0, 1)$.

Define $F(x) = 2\int_0^x f(t)\,dt - [f(x)]^2$ for $x \in [0,1]$.

(a) Show that $F'(x) > 0$ for $x \in (0,1)$.

(b) Show that $\int_0^1 f(t)\,dt > \frac{1}{8}$.

5. Let $u : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function satisfying the following conditions:

- $u''(x) = -u(x)$ for all $x \in \mathbb{R}$;
- $u(0) = 0$;
- $u'(0) = 1$.

Define $v(x) = u(x) - \sin x$ for all $x \in \mathbb{R}$.
By differentiating $w(x) = [v(x)]^2 + [v'(x)]^2$, prove that $u(x) = \sin x$ for all $x \in \mathbb{R}$.

6. (a) Let f be real-valued function defined on an open interval I and $f''(x) \geq 0$ for $x \in I$.

(i) Let $a, b, c \in I$ with $a < c < b$. Using Mean Value Theorem, show that

$$\frac{f(c) - f(a)}{c - a} \leq \frac{f(b) - f(c)}{b - c}.$$

Hence shwow that

$$f(c) \leq \frac{b - c}{b - a} f(a) + \frac{c - a}{b - a} f(b).$$

(ii) Let $a, b \in I$ with $a < b$ and $\lambda \in (0,1)$, show that

$$a < \lambda a + (1 - \lambda)b < b.$$

Hence show that $f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda) f(b)$.

(b) Let $0 < a < b$. Using (a)(ii), show that

(i) if $p > 1$ and $0 < \lambda < 1$, then

$$[\lambda a + (1 - \lambda)b]^p \leq \lambda a^p + (1 - \lambda)b^p;$$

(ii) if $0 < \lambda < 1$, then $\lambda a + (1 - \lambda)b \geq a^\lambda b^{1-\lambda}$.

7. For any real number x, let $[x]$ denote the greatest integer not greater than x. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{2} & \text{when } x \text{ is an integer,} \\ x - [x] - \dfrac{1}{2} & \text{when } x \text{ is not an integer.} \end{cases}$$

(a) (i) Prove that f is a periodic function with period 1.

(ii) Sketch the graph of $f(x)$, where $-2 \leq x \leq 3$.

(iii) Write down all the real number(s) x at which f is discontinuous.

(b) Define $F(x) = \int_0^x f(t)\,dt$ for all real numbers x.

(i) If $0 \le x \le 1$, prove that $F(x) = \dfrac{x^2 - x}{2}$.

(ii) Is F a periodic function? Explain your answer.

(iii) Evaluate $\int_0^\pi F(x)\,dx$.

8. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Assume that a and b are two distinct real numbers.

(i) Find a constant k (independent of x) such that the function

$$h(x) = f(x) - f(b) - f'(x)(x - b) - k(x - b)^2$$

satisfies $h(a) = 0$. Also find $h(b)$.

(ii) Let I be the open interval with end points a and b. Using Mean Value Theorem and (a)(i), prove that there exists a real number $c \in I$ such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(c)}{2}(b - a)^2.$$

(b) Let $g\mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Assume that there exists a real number $\beta \in (0, 1)$ such that $g(x) \le g(\beta) = 1$ for all $x \in (0, 1)$.

(i) Using (a)(ii), prove that there exists a real number $\gamma \in (0, 1)$ such that $g(1) = 1 + \dfrac{g''(\gamma)}{2}(1 - \beta)^2$.

(ii) If $g''(x) \ge -2$ for all $x \in (0, 1)$, prove that $g(0) + g(1) \ge 1$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the following conditions:

- $f(x + y) = e^x f(y) + e^y f(x)$ for all $x, y \in \mathbb{R}$;
- $\lim_{h \to 0} \dfrac{f(h)}{h} = 2005$.

(a) Find $f(0)$.

(b) Find $\lim_{h \to 0} f(h)$. Hence prove that f is a continuous function.

(c) (i) Prove that f is differentiable everywhere and that

$$f'(x) = 2005e^x + f(x)$$

for all $x \in \mathbb{R}$.

(ii) Let n be a positive integer. Using (c)(i), find $f^{(n)}(0)$.

(d) By considering the derivative of the function $\dfrac{f(x)}{e^x}$, find $f(x)$.

10. Let $I_m = \int_0^{\pi/2} \cos^m t\,dt$ where $m = 0, 1, 2, \ldots$.

(a) (i) Evaluate I_0 and I_1.

(ii) Show that $I_m = \dfrac{m-1}{m} I_{m-2}$ for $m \geq 2$.
Hence, evaluate I_{2n} and I_{2n+1} for $n \geq 1$.

(b) Show that $I_{2n-1} \geq I_{2n} \geq I_{2n+1}$ for $n \geq 1$.

(c) Let $A_n = \dfrac{1}{2n+1}\left[\dfrac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right]^2$ where $n = 0, 1, 2, \ldots$.

(i) Using (a) and (b), show that $\dfrac{2n+1}{2n} A_n \geq \dfrac{\pi}{2} \geq A_n$.

(ii) Show that $\{A_n\}$ is monotonic increasing.

(iii) Evaluate $\displaystyle\lim_{n\to\infty} \frac{1}{2n+1}\left[\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right]^2$.

11. (a) Let f be a non-negative continuous function on $[a, b]$. Define

$$F(x) = \int_a^x f(t)\, dt$$

for $x \in [a, b]$.
Show that F is an increasing function on $[a, b]$.
Hence deduce that if $\displaystyle\int_a^b f(t)\, dt = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

(b) Let g be a continuous function on $[a, b]$. If $\displaystyle\int_a^b g(x)u(x)\, dx = 0$ for any continuous function u on $[a, b]$, show that $g(x) = 0$ for all $x \in [a, b]$.

(c) Let h be a continuous function on $[a, b]$. Define

$$A = \frac{1}{b-a}\int_a^b h(t)\, dt.$$

(i) If $v(x) = h(x) - A$ for all $x \in [a, b]$, show that $\displaystyle\int_a^b v(x)\, dx = 0$.

(ii) If $\displaystyle\int_a^b h(x)w(x)\, dx = 0$ for any continuous function w on $[a, b]$ satisfying $\displaystyle\int_a^b w(x)\, dx = 0$, show that $h(x) = A$ for all $x \in [a, b]$.

12. Let n be a positive integer.

(a) Show that $\dfrac{1}{1-t^2} = (1 + t + t^2 + \cdots + t^{2n-2}) + \dfrac{t^{2n}}{1-t^2}$ for $t^2 \neq 1$.

(b) For $-1 < x < 1$, show that

(i) $\displaystyle\int_0^x \frac{t}{1-t^2}\, dt = \ln \frac{1}{\sqrt{1-x^2}}$.

(ii) $\displaystyle\int_0^x \frac{t^{2n+1}}{1-t^2}\, dt = \ln \frac{1}{\sqrt{1-x^2}} - \left(\frac{x^2}{2} + \frac{x^4}{4} + \cdots + \frac{x^{2n}}{2n}\right)$.

(c) Show that $0 \leq \ln 3 - \sum_{k=1}^{n} \frac{1}{2k}\left(\frac{8}{9}\right)^k \leq \frac{9}{2n+2}\left(\frac{8}{9}\right)^{n+1}$.

Hence evaluate $\sum_{k=1}^{\infty} \frac{1}{2k}$.

13. (a) Let $f : (-1,1) \to \mathbb{R}$ be a function with derivatives of any order. For each $m = 1, 2, 3, \ldots$ and $x \in (-1,1)$, define

$$I_m = \frac{1}{(m-1)!}\int_0^x (x-t)^{m-1} f^{(m)}(t)\,dt.$$

(i) Prove that $I_{m+1} = I_m - \frac{f^{(m)}(0)}{m!}x^m$.

(ii) Using (a)(i), prove that

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!}x^k + I_m,$$

where $f^{(0)} = f$.

(b) Define $g(x) = \frac{1}{\sqrt{1-x^2}}$ for all $x \in (-1,1)$. Let n be a positive integer.

(i) Prove that

$$(1-x^2)g'(x) - xg(x) = 0.$$

Hence deduce that

$$(1-x^2)g^{(n+1)}(x) - (2n+1)xg^{(n)}(x) - n^2 g^{(n-1)}(x) = 0,$$

where $g^{(0)}(x) = g(x)$.

(ii) Prove that $g^{(2n-1)}(0) = 0$ and $g^{(2n)}(0) = \left(\frac{(2n)!}{(2^n)(n!)}\right)^2$.

(iii) Using (a), prove that

$$g(x) = \sum_{k=0}^{n-1} \frac{C_k^{2k}}{2^{2k}}x^{2k} + \frac{1}{(2n-1)!}\int_0^x (x-t)^{2n-1} g^{(2n)}(t)\,dt.$$

5 Answers

1. Preliminaries
Section 1.1: Functions

1. $x > 4$ or $x < -2$

2. Not a graph of a function, because there are values of x that correspond to more than one value of y.

3. $x = 0$ or $x = 1$

4. (a) False
   (b) False
   (c) False
   (d) False
   (e) False
   (f) False
   (g) False

Section 1.2: Limits and Derivatives

1. (a) $\frac{1}{24}$ (b) 4 (c) $\frac{1}{4}$ (d) $\frac{1}{e^2}$ (e) $\frac{1}{8}$

2. (a) $3x^2 - 4$ (b) $\frac{1}{\sqrt{x}}$ (c) $-\frac{3}{x^3}$ (d) $-\frac{1}{2x^{\frac{3}{2}}}$

3. (a) $f'(x) = \dfrac{\pi x^{\pi-1}}{x^\pi + e^\pi}$
   (b) $g'(x) = -\dfrac{\pi^x \ln \pi + 4x^3}{2(\pi^x + x^4)^{3/2}}$
   (c) $h'(x) = e^{1/x^2} - \dfrac{2e^{1/x^2}}{x^2} + \dfrac{1 - \ln(2x)}{2x^{3/2}}$
   (d) $u'(x) = \dfrac{4e^{-x} - 2xe^{-x} + 5\sqrt{x}}{4(e^{-x} + \sqrt{x})^{3/2}}$
   (e) $v'(x) = \dfrac{174x}{1 + x^2} - \dfrac{2}{x}$

4. $\dfrac{dy}{dx} = \dfrac{1}{\ln[(x+1)(x+2)^2(x+3)^3]} \left(\dfrac{1}{x+1} + \dfrac{2}{x+2} + \dfrac{3}{x+3} \right)$

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6. (a) False

(b) False
(c) False
(d) False
(e) False

Section 1.3: Applications of Derivatives

1. $(-1, 4)$ or $(3, 12)$

2. $a = -3$, $b = 3$, $c = 0$

3. $\dfrac{1}{9\pi}$ meters per hour

4. (a) The equation of the tangent T_P to C at P is $y = 2$.
   The equation of the normal N_P to C at P is $x = 1$.
   (b) The equation of the tangent T_Q to C at Q is $y = -3x + 4$.
   $(x, y) = (0, 4)$ satisfies the equation of T_Q.

5. (a) $V = \dfrac{\pi h^3}{9}$
   (b) The water level is falling at a rate of $\dfrac{3}{16}$ cm per secondwhen the depth of water is 4 cm.

6. (a) The y-intercept of C is $-\dfrac{4}{3}$.
   The x-intercepts of C are $-\dfrac{2}{3}$, 2.
   (b) $\dfrac{dy}{dx} = 1 - \dfrac{1}{(1+x)^2}$.
   $\dfrac{d^2y}{dx^2} = \dfrac{2}{(1+x)^3}$.
   (c) The turning points of C are $\left(-2, -\dfrac{16}{3}\right)$ and $\left(0, -\dfrac{4}{3}\right)$.
   $\left(-2, -\dfrac{16}{3}\right)$ is a relative maximum.
   $\left(0, -\dfrac{4}{3}\right)$ is a relative minimum.
   (d) The asymptotes of the curve C are the lines $x = -1$, $y = x - \dfrac{7}{3}$
   (e)

7. The function $f(x) = \dfrac{x^2 + kx + 9}{x^2 + 1}$, where k is a constant, attains a stationary value at $x = 3$.

(a) $k = -6$.

(b) i. The x-intercept is 3. The y-intercept is 9.

ii. The turning points of the curve C are $\left(-\frac{1}{3}, 10\right)$, $(3, 0)$.

$\left(-\frac{1}{3}, 10\right)$ is a relative maximum. $(3, 0)$ is a relative minimum.

iii. The only asymptote of C is $y = 1$.

(c)

8. Let Σ be a sphere with (fixed) radius R.

(a) Suppose Γ be a right circular cylinder inscribed in the sphere Σ, with height h, radius r, volume V and surface area A.

i. $V = 2\pi r^2\sqrt{R^2 - r^2}$.

ii. $A = 2\pi r^2 + 4\pi r\sqrt{R^2 - r^2}$.

(b) The largest possible value of V is $\dfrac{4\pi R^3}{3\sqrt{3}}$.

(c) The largest possible value of A is $3\pi R^2$.

9. Let $\beta \in (1, +\infty)$. Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^\beta + \beta - 1 - \beta x$ for any $x \in (0, +\infty)$.

(a) i. $f'(x) = \beta x^{\beta-1} - \beta$ for any $x \in (0, +\infty)$.

ii. Suppose $0 < x < 1$. Then $0 < x^{\beta-1} < 1$. Therefore $f'(x) = \beta(x^{\beta-1} - 1) < 0$.

Hence f is strictly decreasing on $(0, 1]$.

iii. Suppose $x > 1$. Then $x^{\beta-1} > 1$. Therefore $f'(x) = \beta(x^{\beta-1} - 1) > 0$.

Hence f is strictly increasing on $[1, +\infty)$.

iv. f attains the the minimum at 1 on $(0, +\infty)$, with $f(1) = 0$.

(b) Let $r \in (-1, +\infty)$. $1+r \in (0, +\infty)$. Then $f(1+r) = (1+r)^\beta + \beta - 1 - \beta(1+r) \geq 0$.

Therefore $(1 + r)^\beta \geq 1 + \beta r$.

Section 1.4: Integration

1. (a) $27x - 9x^3 + \frac{9}{5}x^5 - \frac{1}{7}x^7 + C$

(b) $\frac{625}{3}x^3 - 125x^4 + 30x^5 - \frac{10}{3}x^6 + \frac{1}{7}x^7 + C$

(c) $\frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x} + C$

(d) $4t^2 - \frac{8}{3}t^{\frac{3}{4}} + C$

2. (a) $-\frac{2}{5}\sqrt{2 - 5x} + C$

(b) $\frac{1}{2}e^{2x} - e^x + x + C$

(c) $-\sqrt{1 - x^2} + C$

(d) $\frac{1}{4}(1 + x^3)^{\frac{4}{3}} + C$

(e) $-\frac{1}{2(1+x^2)} + C$

(f) $-\frac{1}{2}e^{-x^2} + C$

(g) $\ln(2 + e^x) + C$

(h) $x - \ln(1 + e^x) + C$

3. (a) $\frac{14}{3}$ (c) $\frac{1}{8}$
   (b) $\frac{1}{3}$ (d) $\frac{1}{3}$

4. (a) $\frac{32}{3}$ (c) 9 (e) $\frac{10}{3}$
   (b) $\frac{125}{6}$ (d) $\frac{5}{6}$ (f) $\frac{9}{2}$

5. (a) An equation of the curve C is given by $y = -x^2 + 4x - 3$.
   (b) $\dfrac{4}{3}$

6. (a) An equation of the curve C is given by $y = \dfrac{x^3}{3} - 2x + 1$.
   (b) $(0, 1)$

7. (a) $a = -1$, $b = 2$.
   (b) $\dfrac{37}{12}$

8. An equation of the curve C is $y = 3x^2 - \dfrac{1}{x} - 2$.

2. Differentiation

Section 2.2: Limits and Continuity

1. (a) 1
   (b) $\frac{3}{4}$
   (c) 2
   (d) 2
   (e) 3
   (f) 4
   (g) $\frac{1}{2}$
   (h) $\frac{1}{4}$
   (i) $\frac{1}{2}$
   (j) $\frac{1}{\sqrt{3}}$
   (k) 1
   (l) 1

2. (a) 0
   (b) $\frac{3}{5}$
   (c) 3
   (d) $\frac{2}{3}$
   (e) 3
   (f) $\frac{5}{4}$
   (g) -2
   (h) -1
   (i) 3
   (j) -2
   (k) 0
   (l) 1
   (m) $\frac{1}{3}$
   (n) 3
   (o) 3
   (p) 0
   (q) 0
   (r) 0

Section 2.3: Derivatives

1. (a) 0
   (b) not differentiable
   (c) 0
   (d) -2
   (e) not differentiable
   (f) 0

2. $a = -1$, $b = \pi$

3. $a = 0$, $b = \frac{1}{2}$

4. (a) $3x^2 - 4$
   (b) $(x-1)/2x^{3/2}$
   (c) $(5x^2 + 2x)e^{5x}$
   (d) $-\sin x \, \ln x + \cos x/x$
   (e) $\cos 2x$
   (f) $(3\sin x - 1)/\cos^2 x$
   (g) $\cot x - x\csc^2 x$
   (h) $10/(x+2)^2$
   (i) $1 - 2/(x+1)^2$
   (j) $(x\cos(x) - \sin x)/x^2$
   (k) $(2x\tan^2 x - \tan x + 2x)/2x^{3/2}$
   (l) $14x(x^2+1)^6$
   (m) $2x^3/\sqrt{x^4+1}$
   (n) $-2x\sin(x^2)$
   (o) $(3x^3 + x^2 + 1)e^{x^3+x}$
   (p) $1/(x\ln x)$
   (q) $\cos(x)e^{\sin x}$
   (r) $1/(x^2+1)^{3/2}$
   (s) $1/\sqrt{x^2+1}$
   (t) $(1+2\sqrt{x})/(4\sqrt{x}\sqrt{x+\sqrt{x}})$
   (u) $\sinh 2x$
   (v) $\frac{\sinh x}{\cosh^2 x} + \sinh x$
   (w) $\coth x$
   (x) $1/\left(2\sqrt{x(1-x)}\right)$
   (y) $-x/(x^2+1)^{3/2}$

5.

(a) $3^x \ln 3$

(b) $-\ln 2 \sin(x) 2^{\cos(x)}$

(c) $(\ln(x)+1)x^x$

(d) $x^{\sqrt{x}-1/2}(\ln(x)/2+1)$

(e) $(\cos x)^{\sin x - 1}(\cos^2 x + \ln(\cos x)\cos^2 x - 1)$

(f) $x^x x^{x^x}(x(\ln x)^2 + x\ln x + 1)/x$

6. (a) $-\dfrac{x}{y}$

(b) $-\dfrac{y^2+3x^2y}{x^3+2xy}$

(c) $\dfrac{3x^2-2y}{2x-3y^2}$

(d) $-\dfrac{1}{x^2}$

(e) $\dfrac{2x+2y}{x\cos(xy)-2y-2x}$

(f) $\dfrac{\frac{y\sin(y/x)}{x^2}-\frac{1}{x+y}}{\frac{1}{x+y}+\frac{\sin(y/x)}{x}}$

7. (a) $x^{-\frac{3}{2}}e^{x^2} + x^{\frac{1}{2}}e^{x^2} + 4x^{\frac{5}{2}}e^{x^2}$

(b) $-3x(1+x^2)^{-\frac{5}{2}}$

(c) $-\dfrac{2}{x^2}\ln x + \dfrac{2}{x^2}$

(d) $\dfrac{\cos^3 x + 2\sin^2 x \cos x}{\cos^4 x}$

(e) $\dfrac{-2x}{(1+x^2)^2}$

(f) $-\dfrac{8x^2+6y^3}{9y^5}$

(g) $\dfrac{10(2x^6y+3x^4y^3-81y^7)}{x^2(x^2-9y^2)^3}$

(h) $2\cdot 3^{x^2}\ln 3 + 4\cdot x^2 3^{x^2}(\ln 3)^2$

(i) $\dfrac{4y}{x^2}(\ln x)^2 + \dfrac{2y}{x^2}(1-\ln x)$

8. Prove that the Chebyshev polynomials

$$T_m(x) = \frac{1}{2^{m-1}}\cos(m\cos^{-1}x),\ m = 0, 1, 2, \cdots$$

satisfy

$$(1-x^2)T_m''(x) - xT_m'(x) + m^2T_m(x) = 0$$

Proof. By direct computations,

$$T_m'(x) = \frac{m\sin(m\cos^{-1}x)}{2^{m-1}\sqrt{1-x^2}}$$

and

$$T_m''(x) = \frac{m}{2^{m-1}}\left(\frac{x\sin(m\cos^{-1}x)}{(1-x^2)^{\frac{3}{2}}} - \frac{m}{1-x^2}\cos(m\cos^{-1}x)\right).$$

Hence,

$$\begin{aligned}
&(1-x^2)T_m''(x) - xT_m'(x) + m^2T_m(x)\\
=&(1-x^2)\frac{m}{2^{m-1}}\left(\frac{x\sin(m\cos^{-1}x)}{(1-x^2)^{\frac{3}{2}}} - \frac{m}{1-x^2}\cos(m\cos^{-1}x)\right)\\
&-x\frac{m\sin(m\cos^{-1}x)}{2^{m-1}\sqrt{1-x^2}} + m^2\frac{1}{2^{m-1}}\cos(m\cos^{-1}x)\\
=&0.
\end{aligned}$$

□

9. Prove that the Legendre polynomials

$$P_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1)^m \quad m = 0, 1, 2, \cdots$$

satisfy

$$(1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0$$

Proof. Let $g(x) = (x^2-1)^m$, then $P_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}g(x)$.
Because

$$-\frac{d}{dx}(x^2-1)^m = 2mx(x^2-1)^{m-1},$$

therfore,

$$-\frac{d(x^2-1)^m}{dx}(x^2-1) + 2mx(x^2-1)^m = 0.$$

We get

$$\begin{aligned} & -g'(x)(x^2-1) + 2mxg(x) = 0 \\ \Rightarrow \quad & \frac{d^{m+1}}{dx^{m+1}}\left(-g'(x)(x^2-1) + 2mxg(x)\right) = 0. \end{aligned}$$

Apply Leibniz's rule,

$$\begin{aligned} \Rightarrow \quad & -\left(g^{(m+2)}(x)(x^2-1) + C_1^{m+1}g^{(m+1)}(x)\cdot 2x + C_2^{m+1}g^{(m)}(x)\cdot 2\right) \\ & \qquad +2m\left(g^{(m+1)}(x)\cdot x + C_1^{m+1}g^{(m)}(x)\right) = 0 \\ \Rightarrow \quad & (1-x^2)g^{(m+2)}(x) - 2xg^{(m+1)}(x) + m(m+1)g^{(m)}(x) = 0 \\ \Rightarrow \quad & \frac{1}{2^m m!}\left((1-x^2)g^{(m+2)}(x) - 2xg^{(m+1)}(x) + m(m+1)g^{(m)}(x)\right) = 0 \\ \Rightarrow \quad & (1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0. \end{aligned}$$

□

10.

11.

Section 2.4: Mean Value Theorem and Taylor's Theorem

1. Let $f(x) = x^p$ and apply the mean value theorem on $[x, y]$, there exists $c \in (x, y)$ such that

$$\frac{x^p - y^p}{x - y} = pc^{p-1}.$$

Note that $y < c < x$ and $p > 1$, so $y^{p-1} < c^{p-1} < x^{p-1}$. Therefore,

$$py^{p-1} < \frac{x^p - y^p}{x - y} < px^{p-1}.$$

The result follows by multiplying $(x - y)$.

2. Applying mean value theorem, there exist a and b such that $x_1 < a < x_2 < b < x_3$ and
$$\frac{\sin x_2 - \sin x_1}{x_2 - x_1} = \cos a \qquad \text{and} \qquad \frac{\sin x_3 - \sin x_2}{x_3 - x_2} = \cos b.$$
Since $\cos x$ is strictly decreasing on $[0, \pi]$, $\cos a > \cos b$ and the result follows.

3. Let $f(x) = \ln(1+x)$, for $x > 0$, then $f'(x) = \dfrac{1}{1+x}$. By mean value theorem, there exists $c \in (0, x)$ such that
$$\begin{aligned} \frac{\ln(1+x) - \ln 1}{(1+x) - 1} &= \frac{1}{1+c} \\ \ln(1+x) &= \frac{x}{1+c} \end{aligned}$$
Since $0 < c < x$, we have $\dfrac{x}{1+x} < \dfrac{x}{1+c} < x$. Hence
$$\frac{x}{1+x} < \ln(1+x) < x.$$
Since the above inequality holds for all $x > 0$, we can replace x by $\dfrac{1}{x}$ to obtain $\dfrac{1/x}{1+(1/x)} < \ln\left(1+\dfrac{1}{x}\right) < \dfrac{1}{x}$. Therefore,
$$\frac{1}{1+x} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}.$$

4. Let $f(x) = \frac{a_2}{2}x^2 + \frac{a_3}{3}x^3 + \cdots + \frac{a_n}{n+1}x^{n+1}$. Applying mean value theorem, there exists $c \in (0, 1)$ such that
$$\begin{aligned} \frac{f(1) - f(0)}{1 - 0} &= f'(c) \\ \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} &= a_1 c + a_2 c^2 + \cdots + a_n c^n. \end{aligned}$$
Therefore, the equation has a root c in $(0, 1)$.

5. Applying the mean value theorem on $[0, a]$ and $[b, a+b]$, there exist $p \in (0, a)$ and $q \in (b, a+b)$ such that
$$\frac{f(a) - f(0)}{a - 0} = f'(p) \qquad \text{and} \qquad \frac{f(a+b) - f(b)}{(a+b) - b} = f'(q).$$
Since $f'(x)$ is monotonic increasing on $(0, \infty)$, $f'(p) \le f'(q)$. Therefore,
$$\begin{aligned} \frac{f(a)}{a} &\le \frac{f(a+b) - f(b)}{a} \\ f(a) &\le f(a+b) - f(b) \\ f(a) + f(b) &\le f(a+b) \end{aligned}$$

6. Let $x, y \in \mathbb{R}$. The inequality is automatically satisfied if $x = y$, so we only need to consider $x \neq y$. Now, suppose $x > y$. Applying the mean value theorem on $[y, x]$, there exists $c \in (y, x)$ such that

$$\frac{\sin x - \sin y}{x - y} = \cos c \leq 1.$$

Therefore, $|\sin x - \sin y| \leq |x - y|$. Similarly, it holds for the case $y < x$. Thus, $\sin x$ satisfies the Lipschitz condition on R by taking $L = 1$.

7. (a) Applying the mean value theorem on $[k, k+1]$, there exists $c \in (k, k+1)$ such that

$$\frac{f(k+1) - f(k)}{(k+1) - k} = f'(c).$$

Note that, $k < c < k+1$ and $f'(x)$ is strictly decreasing for $x > 0$, so $f'(k+1) < f'(c) < f'(k)$. Therefore,

$$f'(k+1) < f(k+1) - f(k) < f'(k).$$

(b) By the result in (a), for $k = 1, 2, \ldots, n$,

$$f'(k+1) < f(k+1) - f(k) < f'(k).$$

Therefore, $\sum_{k=1}^{n} f'(k+1) < \sum_{k=1}^{n} (f(k+1) - f(k)) < \sum_{k=1}^{n} f'(k)$. Simplify this inequality, we have the result

$$f'(2) + f'(3) + \cdots + f'(n) < f(n) - f(1) < f'(1) + f'(2) + \cdots + f'(n-1).$$

8. (a) $x - \frac{x^2}{2} + \frac{x^3}{3}$

(b) $1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720}$

(c) $x + \frac{x^3}{6} + \frac{x^5}{120}$

9. (a) $\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n!} (x - \frac{\pi}{2})^{2n} = 1 - \frac{1}{2!}(x - \frac{\pi}{2})^2 + \frac{1}{4!}(x - \frac{\pi}{2})^4 - \cdots$

(c) $\sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \cdots$

10. (a) $P_3(x) = x + \frac{x^2}{2} + \frac{x^3}{3}$

(b) $\ln 0.99 = f(0.01) \approx P_3(0.01) = 0.01005033$

11. (a) Firstly, since f is odd, $-f(0) = f(-0)$. Therefore, $f(0) = 0$. Then we have,

$$\begin{aligned} -f(x) &= f(-x) \\ -f'(x) &= -f'(-x) \\ f'(x) &= f'(-x) \\ f''(x) &= -f''(-x) \\ -f''(x) &= f''(-x) \end{aligned}$$

As we can see, f' is an even function and f'' is an odd function again. By repeating the above, it can be shown that $f^{(2n)}$ is an odd function and so $f^{(2n)}(0) = 0$ for all positive integers n. The result follows as $a_{2n} = \dfrac{f^{(2n)}(0)}{(2n)!} = 0$.

(b) Similar to (a), what we have to show is $f^{(2n-1)}(0) = 0$ for all positive integers n.

Section 2.5: L'Hopital's Rule

1. (a) $\frac{3}{5}$
(b) 2
(c) 6
(d) $\frac{1}{3}$
(e) $\frac{1}{3}$
(f) 4
(g) $\frac{1}{2}$
(h) $\frac{1}{2}$
(i) 1
(j) $\frac{e}{2}$
(k) $\ln 2$
(l) e
(m) $\frac{1}{e}$
(n) $\frac{3}{2}$
(o) 1
(p) 1
(q) 3
(r) e

3. Integration
Section 3.1: Fundamental Theorem of Calculus

1. (a) $\dfrac{\cos x}{x}$
   (b) $e^{\sin 2x}$
   (c) $-\sqrt{1+x^2}$
   (d) $3x^2 e^{x^6}$
   (e) $\cos x \cos(\sin^2 x)$
   (f) $2(\ln 2x)^2 - (\ln x)^2$
   (g) $3x^2 e^{\cos(x^3)} - 2x e^{\cos(x^2)}$
   (h) $\dfrac{\sin\sqrt{x}}{x\sqrt{\ln x}}$

Section 3.2: Substitution

1. $-\frac{2}{5}\sqrt{2-5x}+C$
2. $\frac{1}{2}e^{2x}-e^x+x+C$
3. $-\sqrt{1-x^2}+C$
4. $\frac{1}{4}(1+x^3)^{\frac{4}{3}}+C$
5. $-\frac{1}{2(1+x^2)}+C$
6. $2\tan^{-1}\sqrt{x}+C$
7. $\cos\frac{1}{x}+C$
8. $-\frac{1}{2}e^{-x^2}+C$
9. $\ln(2+e^x)+C$
10. $\tan^{-1}e^x+C$
11. $-\ln|\cos x|+C$
12. $x-\ln(1+e^x)+C$
13. $\frac{1}{200}(x^2+2)^{100}+C$
14. $-\sqrt{25-x^2}+C$
15. $\frac{1}{3}\sqrt{3x^2+1}+C$
16. $-\frac{2}{3}\sqrt{9-x^3}+C$
17. $\frac{(x+2)^{101}}{101}-\frac{(x+2)^{100}}{50}+C.$
18. $\frac{1}{12}(2x-5)\sqrt{4x+5}$
19. $\frac{2}{15}(x-1)^{3/2}(3x+2)$
20. $\frac{2}{5}(x-1)^{3/2}(x+4)$
21. $\frac{2}{3}(x-18)\sqrt{x+9}$
22. $\frac{1}{135}\left(3x^2+1\right)^{3/2}\left(9x^2-2\right)$

Section 3.3: Integration by Parts

1. $x\ln x-x+C$
2. $\frac{x^3}{3}(\ln x-\frac{1}{3})+C$
3. $-\frac{1}{x}((\ln x)^2+2\ln x+2)+C$
4. $-(x+1)e^{-x}+C$
5. $-\frac{e^{-2x}}{4}(2x^2+2x+1)+C$
6. $x\sin x+\cos x+C$
7. $-\frac{2x^2-1}{4}\cos 2x+\frac{x}{2}\sin 2x+C$
8. $x\sin^{-1}x+\sqrt{1-x^2}+C$
9. $-\frac{x}{2}+\frac{1+x^2}{2}\tan^{-1}x+C$
10. $x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}+C$
11. $\frac{x^2}{4}-\frac{x}{4}\sin 2x-\frac{1}{8}\cos 2x+C$
12. $\frac{x}{2}(\sin(\ln x)-\cos(\ln x))+C$
13. $\frac{1}{16}\sin(4x)-\frac{1}{4}x\cos(4x)+C$
14. $\cos(5-x)-x\sin(5-x)+C$
15. $x\cos^{-1}(x)-\sqrt{1-x^2}+C$
16. $\frac{1}{4}\left(-x\sqrt{1-x^2}+2x^2\cos^{-1}(x)+\sin^{-1}(x)\right)+C$
17. $x\tan^{-1}(x)-\frac{1}{2}\log(x^2+1)+C$
18. $\frac{1}{2}x^2\tan^{-1}(x)-\frac{x}{2}+\frac{1}{2}\tan^{-1}(x)+C$

19. $2x + x\ln^2(x) - 2x\ln(x) + C$
20. $\frac{1}{100}x^{100}\log(x) - \frac{x^{100}}{10000} + C$
21. $-\frac{1}{10000x^{100}} - \frac{\log(x)}{100x^{100}} + C$
22. $x\tan(x) + \ln(\cos(x)) + C$
23. $\frac{1}{13}e^{2x}(3\sin(3x) + 2\cos(3x)) + C$.

Section 3.5: Trigonometric Integrals

1. $-\cot\frac{x}{2} + C$
2. $\frac{1}{6}\sin^6 x + C$
3. $\frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C$
4. $3\sin\frac{x}{6} + \frac{3}{5}\sin\frac{5x}{6} + C$
5. $\sin x - \frac{1}{3}\sin^3 x + C$
6. $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$
7. $-\frac{1}{\sin x} + \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + C$
8. $-\frac{1}{2}\cos^2 x + \frac{1}{2}\ln(1+\cos^2 x) + C$
9. $\frac{\tan^4}{4} - \frac{\tan^2 x}{2} - \ln|\cos x| + C$
10. $-8\cot 2x - \frac{8}{3}\cot^3 2x + C$
11. $-\frac{1}{8}\cos 4x - \frac{1}{12}\cos 6x + C$
12. $\frac{x}{4} + \frac{\sin 2x}{8} + \frac{\sin 4x}{16} + \frac{\sin 6x}{24} + C$
13. $\frac{\cos^8(x)}{8} - \frac{\cos^6(x)}{6} + C$
14. $\frac{\sin^9(x)}{9} - \frac{2\sin^7(x)}{7} + \frac{\sin^5(x)}{5} + C$
15. $-\frac{1}{6}\cos^5 x\sin x + \frac{1}{24}\cos^3 x\sin x + \frac{1}{16}\cos x\sin x + \frac{1}{16}x + C$.

Section 3.6: Trigonometric Substitution

1. $x - \tan^{-1} x + C$
2. $\dfrac{x}{\sqrt{1-x^2}} + C$
3. $-\sqrt{1-x^2} + \sin^{-1} x + C$
4. $\dfrac{x}{\sqrt{1+x^2}} + C$
5. $\dfrac{9}{2}\sin^{-1}\dfrac{x}{3} - \dfrac{x}{2}\sqrt{9-x^2} + C$
6. $\ln|x + \sqrt{4+x^2}| + C$
7. $\sqrt{16-x^2}\left(\dfrac{x^3}{4} - 2x\right) + 32\sin^{-1}\left(\dfrac{x}{4}\right) + C$
8. $-\dfrac{\sqrt{x^2+4}}{4x} + C$
9. $\dfrac{x}{\sqrt{4x^2+1}}$
10. $\dfrac{x-1}{\sqrt{2x-x^2}}$

Section 3.7: Rational Functions

1. $-x + \frac{1}{2}\ln|\frac{1+x}{1-x}| + C$
2. $9x - \frac{3}{2}x^2 + \frac{1}{3}x^3 - 27\ln|3+x| + C$
3. $x + \ln(1+x^2) + C$
4. $\frac{1}{4}\ln|\frac{x-1}{x+3}| + C$
5. $\frac{1}{10\sqrt{2}}\ln|\frac{x-\sqrt{2}}{x+\sqrt{2}}| - \frac{1}{5\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$
6. $\frac{1}{x+1} + \frac{1}{2}\ln|x^2-1| + C$
7. $-\frac{5x-6}{x^2-3x+2} + 4\ln|\frac{x-1}{x-2}| + C$
8. $\tan^{-1} x + \frac{5}{6}\ln\frac{x^2+1}{x^2+4} + C$
9. $\frac{1}{2}\tan^{-1} x + \frac{1}{4}\ln\frac{(x+1)^2}{x^2+1} + C$
10. $x^2 + 2\ln|x+1| + 3\ln|x-3| + C$
11. $\tan^{-1} x - \frac{1}{x-1} + \ln\frac{x^2+1}{(x-1)^2} + C$
12. $\frac{1}{2(x^2+1)} + \ln|x| - \frac{1}{2}\ln(x^2+1) + C$

13. $\frac{9}{2}\ln(x-3)-4\ln(x-2)+\frac{1}{2}\ln(x-1)+C$

14. $\frac{1}{4}\ln\left(\frac{x^2}{x^2-2x+2}\right)-\frac{1}{2}\tan^{-1}(1-x)+C$

Section 3.8: t-method

1. $-\frac{\cos x}{2\sin^2 x}+\frac{1}{2}\ln|\tan\frac{x}{2}|+C$
2. $\tan x-\sec x+C$
3. $\frac{1}{\cos x}+\frac{1}{3\cos^3 x}+\ln|\tan\frac{x}{2}|+C$
4. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\left(\frac{x}{2}\right)+1}{\sqrt{3}}\right)+C$
5. $2\sqrt{2}\tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{2}}\right)-x+C$
6. $\frac{1}{2}(x+\ln(\sin x+\cos x+3))-\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{2\tan\left(\frac{x}{2}\right)+1}{\sqrt{7}}\right)+C$

Section 3.9: Piecewise Functions

1. $F(x)+C$ where $F(x)=\begin{cases}2x^2-x+5, & x<1\\ 6\sqrt{x}, & x\geq 1\end{cases}$
2. $F(x)+C$ where $F(x)=\begin{cases}\frac{e^{2x}}{2}, & x<0\\ \frac{2x+\sin 2x+2}{4}, & x\geq 0\end{cases}$
3. $\frac{|x-3|(x-3)}{2}+C$
4. $F(x)+C$ where $F(x)=\begin{cases}-\ln(1-x), & x<0\\ \ln(1+x), & x\geq 0\end{cases}$
5. $F(x)+C$ where $F(x)=\begin{cases}x^3-12x, & x<-2\\ -x^3+12x+32, & -2\leq x<2\\ x^3-12x+64, & x\geq 2\end{cases}$
6. $F(x)+C$ where $F(x)=\begin{cases}x(1-\ln x), & 0<x<1\\ x(\ln x-1)+2, & x\geq 1\end{cases}$